

Time dependent embedding of spherically symmetric Rindler spacetime

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March 22, 2012

Abstract

An anisotropic cosmic fluid with radial heat flux which sources a time dependent Rindler geometry is investigated. Even though its energy density ρ is positive, the radial and transversal pressures are negative and the strong energy condition is not satisfied. The congruence of "static" observers is not geodesic and the heat flux is oriented outward. We computed the Misner-Sharp energy associated to the curved Rindler metric embedded in a spatially flat FLRW universe and found that the Weyl energy is vanishing thanks to the conformally flat form of the spacetime.

Keywords: anisotropic cosmic fluid, MS energy, curved Rindler metric, heat flux.

1 Introduction

The apparently simple question whether the cosmological expansion takes place locally (in microphysics or at the Solar System level) is a very complicated one and is still an unsolved problem. The solution depends upon the model of the universe [1, 2, 3, 4, 5]. The aim is to combine both classes of solutions (cosmological and local) and to find out exact solutions for the gravitational field of a particular system immersed in a cosmological background. However, a simple superposition of solutions is not in total agreement with the nonlinear feature of the theory [6].

McVittie [7] was the 1st researcher taking into account the effect of the cosmic expansion on local gravitational systems (a central mass, for example). To avoid the accretion of the cosmic fluid into the central object, he imposed a constraint on its mass. A more recent metric describing a spherical mass in a cosmological background is the Sultana - Dyer geometry [8]. Their line element is conformal to the Schwarzschild one and the stress tensor corresponds to two non-interacting perfect fluids (one timelike and the other null).

From a different point of view Mannheim [9] (see also [10]) obtained a fourth order Einstein's equations starting with a conformally-invariant Lagrangean - the Weyl tensor squared. These equations admit a static spherically-symmetric vacuum solution that contains, apart from the Schwarzschild term $\text{const.}/r$ a new term proportional to r and, therefore, the metric is no longer asymptotically flat. According to Mannheim, this linearly rising potential term shows that a local matter distribution can actually have a global effect at infinity and so gravitational theories become global.

The paper is organized as follows. Sec. 2 deals with the static form of the spherically-symmetric "Rindler" geometry, taking as the starting metric that one from [11]. In Sec. 3 we pay attention to the conformal time dependent version of the Rindler spacetime and keep track of the influence of the universe expansion on the local accelerating observers. The components of the energy-momentum tensor for such observers and the Misner-Sharp (MS) energy are calculated and analysed. Sec. 4 investigates the same Rindler observer embedded in a conformally-flat de Sitter universe and the consequences are evaluated. The last section sets the conclusions.

Throughout the paper we use geometrical units $c = G = 1$ and the positive signature $(-, +, +, +)$. The Latin indices run from 0 to 3 and the coordinates order is (t, r, θ, ϕ) .

2 The spherically-symmetric Rindler metric

We briefly review in this chapter the so-called "Schwarzschild-Rindler metric" [11], firstly obtained by Mannheim [9] and studied later by Grumiller [10]. Removing the central mass M , we reach the metric viewed, in our opinion, by a congruence of uniformly-accelerated observers located on an expanding sphere

$$ds^2 = -(1 - 2gr)dt^2 + (1 - 2gr)^{-1}dr^2 + r^2d\Omega^2, \quad (2.1)$$

where $g > 0$ is the constant acceleration and $d\Omega^2$ stands for the metric on the unit 2-sphere. in [11] the above (curved) metric has been applied in the interior of a relativistic star. To be a solution of the standard Einstein's equations

$$\bar{G}_{ab} \equiv \bar{R}_{ab} - \frac{1}{2}\bar{g}_{ab}\bar{R}^c_c = 8\pi\bar{T}_{ab} \quad (2.2)$$

one shows that a stress tensor is necessary on the RHS with nonzero components

$$\bar{T}_0^0 = -\bar{\rho} = -\frac{g}{2\pi r}, \quad \bar{p}_r = \bar{T}_1^1 = -\bar{\rho}, \quad \bar{T}_2^2 = \bar{T}_3^3 = \bar{p}_\perp = \frac{1}{2}\bar{p}_r, \quad (2.3)$$

where $\bar{\rho}$ is the energy density of the anisotropic fluid, \bar{p}_r is the radial pressure and \bar{p}_\perp represent tangential pressures. Hence, the spacetime (2.1) is sourced by an anisotropic fluid with negative pressure $\bar{p}_r = -\bar{\rho}$ and $\bar{p}_\perp = -\bar{\rho}/2$. It has an event horizon at $r = 1/2g$ and a true singularity at the origin $r = 0$, as can be seen from the scalar curvature and the Kretschmann scalar expressions

$$\bar{R}^a_a = \frac{12g}{r}, \quad \bar{R}^{abcd}{}_{abcd} = \frac{32g^2}{r^2} \quad (2.4)$$

We note that the Weyl tensor vanishes for the line element (2.1). In other words, it can be written in a conformally flat form. In addition, the trace of the energy-momentum tensor (2.3) is negative ($\bar{T}_a^a = -3g/2\pi r$) and, as a consequence, the strong energy condition is not obeyed (as for the dark energy).

3 Curved Rindler metric and universe expansion

In what follows we embed the geometry (2.1) in a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe written in a conformally flat form. The combined metric appears as

$$ds^2 = a^2(t) \left[-(1 - 2gr)dt^2 + (1 - 2gr)^{-1}dr^2 + r^2 d\Omega^2 \right], \quad (3.1)$$

where $a(t)$ is the scale factor and $r < 1/2g$. We look for the influence of the expansion ($\dot{a} \equiv da/dt > 0$) of the universe on the local evolution of the system of uniformly accelerated observers that find themselves in the spacetime (3.1). Let us noting that, far from the horizon ($r \ll 1/2g$) the line element becomes FLRW but when $a(t) = 1$, the curved Rindler geometry is recovered. Our next task is to impose (3.1) to be an exact solution of the Einstein equations

$$G_{ab} \equiv R_{ab} - \frac{1}{2}g_{ab}R_c^c = 8\pi T_{ab}. \quad (3.2)$$

To reach that purpose, T_b^a must have the nonzero components

$$\begin{aligned} -8\pi T_0^0 = 8\pi\rho &= \frac{3\dot{a}^2}{a^4(1-2gr)} + \frac{4g}{a^2r}, & 8\pi T_1^0 &= \frac{2g\dot{a}}{a^3(1-2gr)^2}, \\ 8\pi T_0^1 &= -\frac{2g\dot{a}}{a^3}, & 8\pi T_1^1 &= 8\pi p_r = \frac{\dot{a}^2 - 2a\ddot{a}}{a^4(1-2gr)} - \frac{4g}{a^2r}, \\ 8\pi T_2^2 &= 8\pi T_3^3 = 8\pi p_\perp &= \frac{\dot{a}^2 - 2a\ddot{a}}{a^4(1-2gr)} - \frac{2g}{a^2r}. \end{aligned} \quad (3.3)$$

The scalar curvature is given by

$$R_a^a = \frac{6\ddot{a}}{a^3(1-2gr)} + \frac{12g}{a^2r} \quad (3.4)$$

Let us observe that R_a^a is divergent both at $r = 0$ and at the horizon $r = 1/2g$. The same is valid for the components of T_b^a from (3.3), excepting T_0^1 and T_1^0 which are finite at $r = 0$. When $a(t) = 1$ the expressions (2.3) are recovered. We have always $\rho > 0$ but the signs of p_r and p_\perp depend on the sign of $(\dot{a}^2 - 2a\ddot{a})$. We remark the presence of an energy flux ($T_0^1 \neq 0$) which does not depend on the radial coordinate and becomes null when $g = 0$. In other words, the acceleration g generates a time dependent flux of energy.

Let us consider now a congruence of "static" observers with the velocity vector field

$$u^a = \left(\frac{1}{a(t)\sqrt{1-2gr}}, 0, 0, 0 \right), \quad u^a u_a = -1 \quad (3.5)$$

in the spacetime (3.1). One may check that the above congruence is not geodesic, the acceleration 4-vector being given by

$$a^b \equiv u^a \nabla_a u^b = \left(0, -\frac{g}{a^2}, 0, 0 \right), \quad (3.6)$$

with $\sqrt{a^b a_b} = g/a\sqrt{1-2gr}$. The radial component of (3.6) does not depend on r but the invariant acceleration diverges at the horizon. If one defines formally a surface gravity κ as in the static geometries, one obtains

$$\kappa = \sqrt{a^b a_b} \sqrt{-g_{00}}|_{r=1/2g} = g. \quad (3.7)$$

However, we have to keep in mind that the geometry (3.1) is singular at $r = 1/2g$ and, therefore, the physical meaning of κ is very doubtful. Moreover, as the radial acceleration $a^r < 0$, the force needed to keep the static observer at $r = \text{const.}$ is directed inward. In other words, the gravitational field is repulsive, as expected for a fluid with negative pressures.

As far as the scalar expansion of the congruence is concerned, we have

$$\Theta \equiv \nabla_a u^a = \frac{3\dot{a}}{a^2\sqrt{1-2gr}}, \quad (3.8)$$

which is positive and diverges at the horizon. Its time evolution acquires the form

$$\dot{\Theta} \equiv u^a \nabla_a \Theta = \frac{3(-2\dot{a}^2 + a\ddot{a})}{a^4(1-2gr)}. \quad (3.9)$$

Taking into account the radial and transversal pressures are not equal and $T^1_0 \neq 0$, the source of the metric (3.1) is given by an anisotropic fluid with heat flux

$$T_{ab} = (p_\perp + \rho)u_a u_b + p_\perp g_{ab} + (p_r - p_\perp)s_a s_b + u_a q_b + u_b q_a, \quad (3.10)$$

where $s^a = (0, \sqrt{1-2gr}/a, 0, 0)$ is a spacelike vector orthogonal to u^a , the energy density $\rho = T_{ab}u^a u^b$ and the heat flux is given by

$$q^a = -T^a_b u^b - \rho u^a, \quad q_a u^a = 0. \quad (3.11)$$

Eq. (3.11) yields

$$q^a = \left(0, \frac{g\dot{a}}{4\pi a^4\sqrt{1-2gr}}, 0, 0 \right), \quad (3.12)$$

with $q^r > 0$ and $q \equiv \sqrt{q^a q_a} = g\dot{a}/4\pi a^3(1-2gr)$. It is worth noting that q is finite at $r = 0$ but infinite at $r = 1/2g$. A comparison with Eq. (3.4) from [12] shows that we have here $q^r > 0$, i.e., the flux is oriented outward as if it were emanating from the central singularity. In addition, the constant acceleration g

from (3.12) plays the role of the Newtonian acceleration m/r^2 from Eq. (3.4) of [12]. Since the metric (2.1) is not asymptotically flat, q^r grows when r increases and diverges at $r = 1/2g$.

In order to find energy W that flows across a surface Σ of constant r for an arbitrary time interval, we have to integrate the heat flux (3.11), to obtain

$$W = \int (-T^a_b u^b - \rho u^a) n_a \sqrt{-\gamma} d\theta d\phi dt, \quad (3.13)$$

where $n_a = (0, a/\sqrt{1-2gr}, 0, 0)$ is the normal to the hypersurface of constant r and $\sqrt{-\gamma} = a^3 \sqrt{1-2gr} r^2 \sin\theta$, with γ the determinant of the metric induced on the hypersurface Σ . Using now the expression of T^1_0 from (3.3) and u^a from (3.5), the last equation yields

$$W = -\frac{gr^2}{\sqrt{1-2gr}} \int_{t_2}^{t_1} \dot{a}(t) dt = -\frac{gr^2}{\sqrt{1-2gr}} \Delta a(t) \quad (3.14)$$

with $\Delta a(t) = a(t_2) - a(t_1)$. One observes that $W < 0$, namely the inward energy crossing a $r = \text{const.}$ surface is negative, as expected for a stress tensor that does not satisfy the strong energy condition (*SEC*). We also note that $m(r) = gr^2$ plays the role of a mass as one already remarked in [12]. For a small $\Delta t = t_2 - t_1$, $\Delta a(t)$ is also small and W as well. However, an observer located near $r = 1/2g$ could measure a large W , as if Δt were large.

4 Misner - Sharp energy

The Misner-Sharp (MS) quasilocal energy $E(t, r)$ [2, 13, 14, 6], with its Weyl (E_W) and Ricci (E_R) parts, is useful to detect localized sources of gravity. In the case of spherical symmetry, E is obtained from [13, 6, 15]

$$1 - \frac{2E(t, r)}{r} = g^{ab} R_{,a} R_{,b}, \quad (4.1)$$

where $R = a(t)r$ is the areal radius and $R_{,a} = \partial R / \partial x_a$. One finds, in the spacetime (3.1), that

$$E(t, r) = agr^2 + \frac{\dot{a}^2 r^3}{2a(1-2gr)} \quad (4.2)$$

From the fact that the metric (2.1) (and therefore (3.1)) can be written in a conformally flat form, the Weyl tensor should vanish for the line element (3.1). Therefore, we get $E_W = 0$ [6] because it is proportional to the Weyl scalar. In conclusion, E_R must be given by (4.2). To show this, we remember that our fluid is not perfect and E_R does not depend on ρ only. One obtains, indeed, from

$$E_R = \frac{4\pi}{3} R^3 (\rho - p_r + p_\perp), \quad (4.3)$$

and using the expressions from (3.3) for ρ, p_r and p_\perp , that

$$E_R = \frac{a^3 r^3}{6} \left(\frac{3\dot{a}^2}{a^4(1-2gr)} + \frac{6g}{a^2 r} \right) \quad (4.4)$$

which is exactly (4.2). One also observes that E_R from (4.3) acquires the form $E_R = (4\pi/3)R^3\rho$ when we are dealing with a perfect fluid ($p_r = p_\perp$).

It is obvious that $E_W = 0$ (we have no a mass term in the metric). However, let us note that the 1st term on the RHS of (4.2) may be interpreted as a mass term if we write it as $a(t)m(r)$, with $m(r) = gr^2$ - a form already proposed in [11] for the metric (2.1).

5 Time dependent Rindler - de Sitter geometry

Our next task is to consider a particular choice of the scale factor $a(t)$. As in the previous paper [12] we choose the conformally flat de Sitter geometry as the space where the metric (2.1) is embedded, for to preserve the causal structure. Therefore, we take

$$ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + dr^2 + r^2 d\Omega^2), \quad (5.1)$$

where η is the conformal time and H is the Hubble constant. η is related to the cosmic time by $H\eta = -e^{-H\bar{t}}$ ($\bar{t} > 0$, $-1/H < \eta < 0$) [12]. The "combined" spacetime appears as

$$ds^2 = \frac{1}{H^2\eta^2} \left[-(1-2gr)d\eta^2 + \frac{dr^2}{1-2gr} + r^2 d\Omega^2 \right], \quad (5.2)$$

where $a(\eta) = 1/H|\eta|$. Using this special value of the scale factor, we may write down the expression of the energy density and pressures of the anisotropic fluid

$$8\pi\rho = -8\pi p_r = \frac{3H^2}{1-2gr} + \frac{4gH^2\eta^2}{r}, \quad 8\pi p_\perp = -\frac{3H^2}{1-2gr} - \frac{2gH^2\eta^2}{r} \quad (5.3)$$

It is easy to observe that the fluid becomes isotropic ($p_r = p_\perp \equiv p$) when $g = 0$ and, in addition, $\rho = -p = 3H^2$, as it should be for a de Sitter universe. Moreover, the isotropy is also obtained when $\eta \rightarrow 0$ (or $\bar{t} \rightarrow \infty$). Nevertheless, in the latter case $\rho = -p = 3H^2/(1-2gr)$. We observe from (5.3) that the energy density and pressures become infinite at $r = 0$ or at the event horizon $r = 1/2g$. It is worth noting the positivity of ρ for any values of $-1/H < \eta < 0$ and $r < 1/2g$. In contrast, p_r and p_\perp are always negative. The results are here more realistic than those obtained in [16], where ρ , p_r and p_\perp change their sign for certain values of r or η . Therefore, the weak energy condition is satisfied in the present case.

The scalar curvature, expansion and acceleration of the congruence are

$$\Theta = \frac{3H}{1-2gr}, \quad R^a_a = \frac{12H^2}{1-2gr} + \frac{12gH^2\eta^2}{r}, \quad a^b = (0, -gH^2\eta^2, 0, 0) \quad (5.4)$$

The expansion Θ is time independent ($\dot{\Theta} = 0$) and the curvature acquires the pure de Sitter value $12H^2$ when $g = 0$. For the heat flux one obtains

$$q^r = \frac{gH^3\eta^2}{4\pi\sqrt{1-2gr}}, \quad \sqrt{q^a q_a} = \frac{gH^2|\eta|}{4\pi(1-2gr)} \quad (5.5)$$

For instance, taking the cosmic time $\bar{t} \ll H^{-1}$ (or $H|\eta| \approx 1$) and with $g \cong 10^3 \text{ cm/s}^2$, we have

$$q^r \approx \frac{c^2}{G} \frac{gH}{4\pi} = 10^{12} \text{ erg/cm}^2 \text{ s} \quad (5.6)$$

We have used here $H \approx 10^{-18} \text{ s}$ and $r \ll 1/2g$. The expression (5.6) resembles Eq. (4.10) from [12] where instead of g we have m/r^2 , m being the Schwarzschild mass. The constant flux (5.6) is outgoing ($q^r > 0$) and equals its value at the origin $r = 0$.

6 Conclusions

We studied in this paper the (curved) Rindler space embedded in a spatially flat FLRW universe. In order to be an exact solution of Einstein's equations, the "combined" metric must be sourced by an anisotropic fluid with negative pressures and nonvanishing heat flux. The scalar invariants diverges at the origin of coordinates and at the horizon $r = 1/2g$, g being the constant acceleration of a series of radially expanding observers (from the point of view of inertial observers). The off diagonal components of the stress tensor become null when $g = 0$, that is the energy flux is generated by nonzero acceleration. Because of the anisotropy of the fluid, the Ricci part of the MS energy depends not only on ρ but also on the pressures (the Weyl part vanishes thanks to the conformality of the metric). We finally studied the curved Rindler geometry embedded in a de Sitter space written in a conformally flat form. One obtains now a time independent scalar expansion of the congruence and for $g = 0$ the equation of state of the fluid acquires the form $\rho = -p = 3H^2$. So the pure de Sitter values are recovered.

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